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Spatially Homogeneous World Models*

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We obtain the field equations of Einstein for spatially homogeneous spaces as the Euler–Lagrange equations of a variational problem. We write these equations in Hamiltonian form and regularize them. In this way, we obtain a class of solutions without rotations. We derive, in particular, the Lagrangian function for the rotating model with the S^3 group first computed by Gödel. We suggest that the corresponding Hamiltonian equations can be regularized.

1. INTRODUCTION

It is the main objective of some cosmologists nowadays to treat the following outstanding problem of the relativistic cosmology: Consider the line element

$$ds^2 = dt^2 + A(\omega^1)^2 + B(\omega^2)^2 + C(\omega^3)^2 + 2D\omega^2\omega^3, \quad (1.1)$$

where ω^1, ω^2, and ω^3 are the invariant differential forms of the group S^3 satisfying the relations

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^3 \wedge \omega^1,$$
$$d\omega^3 = -\omega^1 \wedge \omega^2, \quad (1.2)$$

and A, B, C, and D are functions of t only. We find the solution in form (1.1) to Einstein's field equations with dust such that the rotation and the expansion of the matter are different from zero. This is interesting not because the astronomers had discovered the rotation of the universe, but it is interesting from a theoretical point of view. This model probably would be the simplest world model with finite space part and with the most general motion of the "Weltsubstrat," that is, with nonvanishing translation, rotation, expansion, and shear.

We had this problem in mind when developing this paper, the structure of which is as follows: Using an idea of Weyl, we obtain Einstein's field equations for spatially homogeneous spaces as the Euler–Lagrange equations of a variational problem. More precisely, we obtain the vacuum field equations for all the groups and the field equations with incoherent matter for Class I groups only. We call Class I the Bianchi Type

I, II, VIII, and IX groups, characterized by

$$d\omega^1 = 0, \quad d\omega^2 = 0, \quad d\omega^3 = 0, \tag{1.3}$$

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = 0, \quad d\omega^3 = 0, \tag{1.4}$$

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^3 \wedge \omega^1, \quad d\omega^3 = -\omega^1 \wedge \omega^2, \tag{1.5}$$

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^3 \wedge \omega^1, \quad d\omega^3 = -\omega^1 \wedge \omega^2. \tag{1.6}$$

Our reasons for this should be clear later.

We apply the general theory to the line element

$$ds^2 = dt^2 - (A\omega^1)^2 - (B\omega^2)^2 - (B\omega^3)^2 \tag{1.7}$$

with each of the groups (1.3)–(1.6). We write the Euler–Lagrange equations in Hamiltonian form. We regularize these equations; that is, we introduce a new variable τ by a suitable transformation on t such that these equations transform into an *analytic* system. This system is then easily solved by a computer, or one can think of the solution developed into convergent power series with respect to the regularizing parameter τ. As a second application, we derive the Lagrangian function for (1.1) with (1.2) first given by Gödel.[1] It is obvious that there are several ways to regularize the corresponding canonical equations; therefore, we can say that *the problem of the rotating universe can be solved by regularization*, in the same sense as Sundman solved the 3-body problem of the celestial mechanics. We do not give here, however, any explicit regularizing transformation, since there might be a "much better" one than the obvious one.

In closing, we refer to a remarkable talk delivered by Misner at the Cincinnati Conference.[2] Misner specializes the Arnowitt, Deser, and Misner formalism to type IX spaces in order to obtain the Einstein equations for

$$ds^2 = dt^2 - (A\omega^1)^2 - (B\omega^2)^2 - (C\omega^3)^2.$$

He writes the field equations in Hamiltonian form, introduces a new parameter instead of time, and discusses the singularities of the model. Our approach is similar. It is based on ideas of Weyl[3] and Gödel,[1] designed for spatially homogeneous spaces, and we think it is simpler. Concerning the introduction of a new parameter instead of the time, we follow Sundman,[4] as explained.

2. PRELIMINARIES

We consider the Lie group $M_4 = R \times G_3$, where R is the real line and G_3 is a 3-dimensional Lie group. Denoting by t the coordinate on R, we can introduce the vector fields

$$X_0 = \frac{\partial}{\partial t}, \quad X_a, \quad a = 1, 2, 3, \tag{2.1}$$

and the 1-forms

$$\omega^0 = dt, \quad \omega^a, \quad a = 1, 2, 3, \tag{2.2}$$

such that

$$\omega^\alpha(X_\beta) = \delta^\alpha{}_\beta, \quad \alpha, \beta \cdots 0, 1, 2, 3, \tag{2.3}$$

and X_a and ω^a are invariant under the left translations of G_3. One knows that the left-invariant vector fields of M_4 form a Lie algebra; that is,

$$[X_0, X_a] = 0, \quad a = 1, 2, 3, \quad [X_a, X_b] = C^f{}_{ab} X_f \tag{2.4}$$

or

$$d\omega^0 = 0, \quad d\omega^a = -\tfrac{1}{2} C^a{}_{bc} \omega^b \wedge \omega^c, \tag{2.5}$$

where $C^a{}_{bc}$ are the components of the structure constant tensor of the Lie algebra of G_3 with respect to the base (2.1) and (2.2). The C's satisfy

$$C^a{}_{bc} = -C^a{}_{cb} \tag{2.6}$$

and the Jacobi identities

$$C^a{}_{fb} C^f{}_{cd} + C^a{}_{fc} C^f{}_{db} + C^a{}_{fd} C^f{}_{bc} = 0. \tag{2.7}$$

We use X_α and ω^α to span the tensor algebra over M_4, that is, we specify tensor fields by giving their components with respect to these bases. We introduce on M_4 a connection by

$$\nabla_{X_\alpha} X_\beta = \Gamma_{\alpha\beta}{}^\gamma X_\gamma, \tag{2.8}$$

where $\Gamma_{\alpha\beta}{}^\gamma$ are the components of the connection with respect to (2.1). The curvature tensor field of the connection is given by

$$R(U, V)Y = \nabla_U \nabla_V Y - \nabla_V \nabla_U Y - \nabla_{[U,V]} Y,$$

and from that we have

$$R(X_\gamma, X_\delta)X_\beta = \nabla_{X_\gamma} \nabla_{X_\delta} X_\beta - \nabla_{X_\delta} \nabla_{X_\gamma} X_\beta - \nabla_{[X_\gamma, X_\delta]} X_\beta = R^\alpha{}_{\beta\gamma\delta} X_\alpha.$$

Following the roles of the covariant differentiation, we compute that

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma_{\gamma\sigma}{}^\alpha \Gamma_{\delta\beta}{}^\sigma - \Gamma_{\delta\sigma}{}^\alpha \Gamma_{\gamma\beta}{}^\sigma - \Gamma_{\sigma\beta}{}^\alpha C^\sigma{}_{\gamma\delta} + X_\gamma \Gamma_{\delta\beta}{}^\alpha - X_\delta \Gamma_{\gamma\beta}{}^\alpha. \tag{2.9}$$

By requiring that the torsion tensor field

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

vanishes, that is,

$$\nabla_{X_\alpha} X_\beta - \nabla_{X_\beta} X_\alpha - [X_\alpha, X_\beta] = 0,$$

we obtain the following symmetry properties for the Γ's:

$$\Gamma_{\alpha\beta}{}^{\gamma} - \Gamma_{\beta\alpha}{}^{\gamma} = C^{\gamma}{}_{\alpha\beta}. \tag{2.10}$$

Using (2.10) we compute the components of the Ricci tensor field:

$$\begin{aligned} R_{\beta\gamma} &= R^{\alpha}{}_{\beta\gamma\alpha} \\ &= \Gamma_{\psi\beta}{}^{\phi}\Gamma_{\phi\gamma}{}^{\psi} - \Gamma_{\psi\phi}{}^{\psi}\Gamma_{\beta\gamma}{}^{\phi} + X_{\beta}\Gamma_{\psi\gamma}{}^{\psi} - X_{\psi}\Gamma_{\beta\gamma}{}^{\psi}. \end{aligned} \tag{2.11}$$

Introducing on M_4 a metric by the requirement that

$$g(X_\alpha, X_\beta) = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_{ab}(t) \end{bmatrix} = \gamma_{\alpha\beta}(t), \quad \alpha, \beta = 0, 1, 2, 3, \tag{2.12}$$

that is,

$$ds^2 = dt^2 + \gamma_{ab}(t)\omega^a\omega^b, \quad a, b = 1, 2, 3, \tag{2.13}$$

M_4 becomes a pseudo-Riemannian space. Equation (2.13) excludes the possibility of lightlike t, but it is at our disposal to choose it timelike or spacelike. The metric (2.13) is left invariant under the transformations of G_3, but not invariant under M_4. We call these metrics spatially homogeneous, meaning that there exist global 3-dimensional hypersurfaces generated by G_3. The name also suggests that these hypersurfaces are spacelike. We make some remarks regarding this point later.

The requirement that the metric (2.13) should be covariant constant with respect to the connection (2.8), combined with the requirement (2.10), leads to the following equations:

$$\begin{aligned} 2g(\nabla_{X_\alpha} X_\beta, X_\gamma) &= X_\alpha g(X_\beta, X_\gamma) + X_\beta g(X_\gamma, X_\alpha) - X_\gamma g(X_\alpha, X_\beta) \\ &\quad + g(X_\beta, [X_\gamma, X_\alpha]) + g(X_\gamma, [X_\alpha, X_\beta]) \\ &\quad - g(X_\alpha, [X_\beta, X_\gamma]), \end{aligned}$$

that is,

$$\begin{aligned} \Gamma_{\alpha\beta\gamma} &\equiv \Gamma_{\alpha\beta}{}^{\sigma}\gamma_{\sigma\gamma} \\ &= \tfrac{1}{2}(X_\alpha\gamma_{\beta\gamma} + X_\beta\gamma_{\gamma\alpha} - X_\gamma\gamma_{\alpha\beta}) \\ &\quad + \tfrac{1}{2}(C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta} - C_{\alpha\beta\gamma}), \end{aligned} \tag{2.14}$$

where

$$C_{\alpha\beta\gamma} = \gamma_{\alpha\sigma}C^{\sigma}{}_{\beta\gamma}. \tag{2.15}$$

Using (2.14), we observe that

$$X_a\gamma_{\alpha\beta} = 0, \quad a = 1, 2, 3, \tag{2.16}$$

and introduce the notation

$$X_0 f = \dot{f}, \tag{2.17}$$

where f is a function over M_4; we can write (2.8) explicitly with the γ's as

$$\begin{aligned} &\nabla_{X_0}X_0 = 0, \qquad \nabla_{X_0}X_a = K_a{}^bX_b, \\ &\nabla_{X_a}X_0 = K_a{}^bX_b, \quad \nabla_{X_a}X_b = -\tfrac{1}{2}\dot{\gamma}_{ab}X_0 + \Gamma_{ab}{}^cX_c, \end{aligned} \tag{2.18}$$

where

$$K_a{}^b = \tfrac{1}{2}\dot{\gamma}_{af}\gamma^{fb} \tag{2.19}$$

and

$$\Gamma_{ab}{}^c = \gamma^{cf}\tfrac{1}{2}(C_{bfa} + C_{fab} - C_{abf}). \tag{2.20}$$

We now substitute (2.18) into (2.11) and find that the components of the Ricci tensor field are given by

$$R_{00} = (K_f{}^f)^{\cdot} + K_f{}^gK_g{}^f, \tag{2.21}$$

$$R_{a0} = K_g{}^f\Gamma_{fa}{}^g - K_a{}^f\Gamma_{gf}{}^g \equiv K_g{}^fC^g{}_{fa} - K_a{}^fC^g{}_{gf}, \tag{2.22}$$

$$R_{ab} = \tfrac{1}{2}\ddot{\gamma}_{ab} - K_a{}^f\dot{\gamma}_{fb} + \tfrac{1}{2}\dot{\gamma}_{ab}(K_f{}^f) + R^*{}_{ab}, \tag{2.23}$$

where

$$R^*{}_{ab} = \Gamma_{fa}{}^g\,\Gamma_{gb}{}^f - \Gamma_{fg}{}^f\,\Gamma_{ab}{}^g \tag{2.24}$$

is the Ricci tensor field of the group space G_3. These expressions have been calculated by Taub[5] and Heckmann and Schücking.[6] Using the identity

$$\tfrac{1}{2}\gamma_{af}\ddot{\gamma}^{fb} = (K_a{}^b)^{\cdot} + 2K_a{}^fK_f{}^b, \tag{2.25}$$

we can compute the Ricci scalar R,

$$\tfrac{1}{2}R = (K_f f)^{\cdot} + \tfrac{1}{2}[K_f{}^gK_f{}^f + (K_f{}^f)^2] + \tfrac{1}{2}R^*, \tag{2.26}$$

where

$$R^* = \gamma^{ab}R^*{}_{ab} \tag{2.27}$$

is the Ricci scalar of the group spaces G_3.

We consider Einstein's field equations with incoherent matter written in the form

$$R_{\alpha\beta} - \tfrac{1}{2}R\gamma_{\alpha\beta} = -\kappa\rho u_\alpha u_\beta, \tag{2.28}$$

$$u_\alpha u^\alpha = 1 \tag{2.29}$$

for the spaces (2.13), where

$$U = u^\alpha X_\alpha = uX_0 + u^aX_a \tag{2.30}$$

and

$$\mu = u_\alpha\omega^\alpha = u\,dt + u_a\omega^a \tag{2.31}$$

are the velocity vector field of the matter and the corresponding 1-form, respectively. The normalization (2.29) chooses the t lines to be timelike. One has to make this choice for physical and not for mathematical reasons. Choosing, instead of (2.29), the normalization

$$u_\alpha u^\alpha = -1, \tag{2.32}$$

we could extend our future discussions to spacelike t lines. The corresponding solutions, however, would represent stationary world models, where the density of the matter is a function of the spacelike coordinate t only. One is not looking systematically for such models without having a special reason. We would like to remark, however, that there are interesting special solutions for spacelike t lines in the case of the Bianchi Type VIII group if we include a nonvanishing Λ term into our discussions. These solutions are given by the line elements

$$\begin{aligned} ds^2 = dt^2 &+ (1-k)(\omega^1 \cos\beta t + \omega^2 \sin\beta t)^2 \\ &+ (1+k)(-\omega^1 \sin\beta t + \omega^2\cos\beta t)^2 \\ &- (1+2k^2)(\omega^3)^2, \end{aligned} \tag{2.33}$$

where

$$\beta = \left(\frac{1-2k^2}{2(1+2k^2)}\right)^{\frac{1}{2}}, \quad \tfrac{1}{2} < |k| \le \frac{1}{2^{\frac{1}{2}}},$$

k a real parameter,

and

$$ds^2 = dt^2 + \tfrac{1}{2}(1+s)(\omega^1)^2 + \tfrac{1}{2}(1-s)(\omega^2)^2 - (\omega^3)^2, \tag{2.34}$$

where

$$|s| < 1, \quad s \text{ a real parameter},$$

and

$$\begin{aligned} d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^3 \wedge \omega^1, \\ d\omega^3 = -\omega^1 \wedge \omega^2, \end{aligned} \tag{2.35}$$

or in a special coordinate system

$$\begin{aligned} \omega^1 &= \cos x^3\, dx^1 = e^{x^1} \sin x^3\, dx^2, \\ \omega^2 &= -\sin x^3\, dx^1 + e^{x^1}\cos x^3\, dx^2, \\ \omega^3 &= e^{x^1}\, dx^2 + dx^3. \end{aligned} \tag{2.36}$$

Equations (2.33) and (2.24) are the Class II and Class III universes discovered by the author.[7] Equation (2.34) contains the famous Gödel cosmos[8] as a special case for $s = 0$. The speciality of (2.33) and (2.34) is that they are invariant under a 4-dimensional Lie group containing (2.35) as an invariant subgroup. As a consequence of that, the density of the matter is constant.

Coming back to our main line of reasoning, we list a few formulas for later use. The components of the tensor fields $\nabla_X U$ and $\nabla_X \mu(Y)$ are

$$u^\alpha{}_{:\beta} = X_\beta u^\alpha + \Gamma_{\beta\gamma}{}^\alpha u^\gamma \quad \text{and} \quad u_{\alpha:\beta} = X_\beta u_\alpha - \Gamma_{\beta\alpha}{}^\gamma u_\gamma, \tag{2.37}$$

as one easily sees following the roles of the covariant differentiation. All our subsequent formulas are consequences of (2.37). The equations of geodesic motion and the continuity equations are

$$u\dot{u} - \tfrac{1}{2}\dot{\gamma}_{fg}u^f u^g = 0, \tag{2.38}$$

$$uu_a - C^g{}_{fa}u^f u_g = 0, \tag{2.39}$$

$$(\rho u)^\cdot + \rho u(K_f{}^f) + \rho C^g{}_{gf}u^f = 0. \tag{2.40}$$

For later references, we write (2.29) as

$$(u)^2 + u_f u^f = 1. \tag{2.41}$$

Equation (2.38) is a consequence of (2.39) and (2.40). The components of the tensor of rotation are

$$\omega_{\alpha\beta} = \tfrac{1}{2}(X_\alpha u_\beta - X_\beta u_\alpha) - \tfrac{1}{2}C^\gamma{}_{\alpha\beta}u_\gamma, \tag{2.42}$$

that is,

$$\omega_{ab} = -\tfrac{1}{2}C^f{}_{ab}u_f, \quad \omega_{a0} = -\tfrac{1}{2}\dot{u}_a. \tag{2.43}$$

3. VARIATIONAL PRINCIPLE FOR VACUUM

Since the vacuum case already contains some essential features of our problem, for the sake of simplicity we start with this case. We write the vacuum field equations using (2.21), (2.22), (2.23), (2.25), and (2.26) in the following form:

$$R_{00} - \tfrac{1}{2}R = \tfrac{1}{2}(K_f{}^g K_g{}^f - (K_f{}^f)^2 - R^*) = 0, \tag{3.1}$$

$$R_{a0} = K_g{}^f C^g{}_{fa} - K_a{}^f C^g{}_{gf} = 0, \tag{3.2}$$

$$\begin{aligned} R_a{}^b - \tfrac{1}{2}R\delta_a{}^b = (K_a{}^b)^\cdot &- (K_f{}^f)\delta_a{}^b + (K_f{}^f)K_a{}^b \\ &- \tfrac{1}{2}[K_f{}^g K_g{}^f + (K_f{}^f)^2]\delta_a{}^b \\ &+ R^*{}_a{}^b - \tfrac{1}{2}R^*\delta_a{}^b = 0. \end{aligned} \tag{3.3}$$

These are Taub's equations[5] written in a slightly different form. Weyl writes in his famous book (Ref. 3, p. 251) while calculating the static spherically symmetric field for vacuum: "Wir nutzen das Wirkungsprinzip zunächst nur teilweise aus, indem wir annehmen, dass bei der Variation die zugrunde gelegte Normalform des ds^2 nicht zerstört wird; · · · bei solcher eingeschränkten Verwendung genügt es, das Wirkungsintegral für jene Normalform zu berechnen." These ideas apply in our case word for word.

The normal form for ds^2 is given in our case by (2.13). The action integral for this normal form is

$$\int \tfrac{1}{2}Rg^{\frac{1}{2}}\, dx = \int_{t_1}^{t_2} \tfrac{1}{2}R^0\gamma^{\frac{1}{2}}\, dt \int_G \omega^1 \wedge \omega^2 \wedge \omega^3, \tag{3.4}$$

where $\frac{1}{2}R$ is given by (2.26) and

$$\gamma = |\det(\gamma_{ab})|. \tag{3.5}$$

The first integral in (3.4) is extended between two fixed values of t, the second one over G_3 if compact, or over a part of it if otherwise, giving a finite constant

C which we normalize later. The action integral is, therefore,

$$J = c\int_{t_1}^{t_2} \gamma^{\frac{1}{2}}\{(K_f{}^f)^{\cdot} + \tfrac{1}{2}[K_f{}^g K_g{}^f + (K_f{}^f)^2] + \tfrac{1}{2}R^*\}\, dt. \tag{3.6}$$

Using the identity

$$(\gamma^{\frac{1}{2}})^{\cdot} = \gamma^{\frac{1}{2}} K_f{}^f \tag{3.7}$$

and integrating by parts and normalizing $c = 2$, we have

$$J = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 + R^*]\, dt. \tag{3.8}$$

But varying (3.8) with respect to the γ's is precisely the requirement to consider variations, which leaves the normal form (2.13) unchanged.

We now prove that

$$\delta J = 0 \tag{3.9}$$

gives the six field equations (3.3) as Euler–Lagrange equations. The proof is a straightforward calculation. We assume, following Siegel,[4] that our variational problem has a solution, and we consider a family of functions

$$\gamma_{ab}(\alpha; t), \quad -1 < \alpha < 1, \tag{3.10}$$

such that

$$\gamma_{ab}(0; t) = \gamma_{ab}(t) \tag{3.11}$$

is the solution of our variational problem. We construct with these functions the integral

$$J(\alpha) = \int_{t_1}^{t_2} L[\gamma_{ab}(\alpha; t), \dot{\gamma}_{ab}(\alpha; t)]\, dt, \quad -1 < \alpha < 1. \tag{3.12}$$

As a consequence of our assumption, $J(\alpha)$ assumes its extremum at $\alpha = 0$ and, therefore,

$$J'\big|_{\alpha=0} \equiv \frac{dJ(\alpha)}{d\alpha}\bigg|_{\alpha=0} = 0 \tag{3.13}$$

is equivalent to (3.9).

Carrying out our calculations, we see that

$$\gamma_{ab}(\alpha; t_1) = a_{ab} = \text{const}, \quad \gamma_{ab}(\alpha; t_2) = b_{ab} = \text{const}, \tag{3.14}$$

and that

$$\gamma'_{ab}(0; t) \text{ is arbitrary.} \tag{3.15}$$

The formula for $dJ/d\alpha$ reads as

$$\begin{aligned} \frac{dJ}{d\alpha} &= \int_{t_1}^{t_2} (\gamma)^{\frac{1}{2}\prime}[K_f{}^g K_g{}^f - (K_f{}^f)^2 + R^*]\, dt \\ &\quad + 2\int_{t_1}^{t_2} \gamma^{\frac{1}{2}}[(K_f{}^g)' K_g{}^f - (K_f{}^f)(K_g{}^g)']\, dt \\ &\quad + \int_{t_1}^{t_2} \gamma^{\frac{1}{2}} (\gamma^{fg})' R^*{}_{fg}\, dt. \end{aligned} \tag{3.16}$$

We know that we do not have to compute $(R^*{}_{ab})'$. In order to facilitate the calculations, we compute the expressions

$$(\gamma^{fg})' = -\gamma^{fa}\gamma'_{ab}\gamma^{bg}, \tag{3.17}$$

$$(\gamma^{\frac{1}{2}})' = \tfrac{1}{2}\gamma^{\frac{1}{2}}\gamma'_{ab}\gamma^{ab}, \tag{3.18}$$

$$(K_f{}^g)' = \tfrac{1}{2}\dot{\gamma}'_{fa}\gamma^{ag} - K_f{}^a\gamma^{bg}\gamma'_{ab}, \tag{3.19}$$

$$(K_f{}^f)' = \tfrac{1}{2}\dot{\gamma}'_{ab}\gamma^{ab} - \gamma^{af}K_f{}^b\gamma'_{ab}. \tag{3.20}$$

Substituting into (3.16), we obtain

$$\begin{aligned} \frac{dJ}{d\alpha} &= \int_{t_1}^{t_2} \gamma^{\frac{1}{2}}(\gamma^{af}K_f{}^b - \gamma^{ab}K_f{}^f)\dot{\gamma}'_{ab}\, dt \\ &\quad - \int_{t_1}^{t_2} \gamma^{\frac{1}{2}}[2\gamma^{af}K_f{}^g K_g{}^b - 2\gamma^{af}K_f{}^b(K_g{}^g) \\ &\quad - \tfrac{1}{2}\gamma^{ab}(K_f{}^g K_g{}^f - K_f{}^f)^2 + R^{*ab} - \tfrac{1}{2}\gamma^{ab}R^*]\gamma'_{ab}\, dt. \end{aligned} \tag{3.21}$$

We now evaluate the first integral in (3.21). Integrating by parts and using (3.7) and (3.14), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} \gamma^{\frac{1}{2}}[\gamma^{af}K_f{}^b - \gamma^{ab}(K_f{}^f)]\dot{\gamma}'_{ab}\, dt \\ &\quad = -\int_{t_1}^{t_2} \gamma^{\frac{1}{2}}[\gamma^{af}(K_f{}^b)^{\cdot} - \gamma^{ab}(K_f{}^f)^{\cdot} - 2\gamma^{af}K_f{}^g K_g{}^b \\ &\quad + 3\gamma^{af}K_f{}^b(K_g{}^g) - \gamma^{ab}(K_f{}^f)^2]\gamma'_{ab}\, dt. \end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.21), we obtain

$$\begin{aligned} \frac{dJ}{d\alpha} &= -\int_{t_1}^{t_2} \gamma^{\frac{1}{2}}[\gamma^{af}(K_f{}^b)^{\cdot} - \gamma^{ab}(K_f{}^f)^{\cdot} + \gamma^{af}K_f{}^b(K_g{}^g) \\ &\quad - \tfrac{1}{2}\gamma^{ab}[K_f{}^g K_g{}^f + (K_f{}^f)^2] \\ &\quad + R^{*ab} - \tfrac{1}{2}R^*\gamma^{ab}]\gamma'_{ab}\, dt. \end{aligned} \tag{3.23}$$

Substituting $\alpha = 0$ and using (3.13) and (3.15), we find that

$$\begin{aligned} &-\gamma^{\frac{1}{2}}\gamma^{la}\{(K_a{}^b)^{\cdot} - (K_g{}^g)^{\cdot}\delta_a{}^b + K_a{}^b(K_g{}^g) \\ &-\tfrac{1}{2}[K_f{}^g K_g{}^f + (K_f{}^f)^2] + R^*{}_a{}^b - \tfrac{1}{2}R^*\delta_a{}^b\} = 0, \end{aligned} \tag{3.24}$$

and a glance at (3.3) proves our assertion. The Lagrangian of the vacuum problem is, therefore,

$$L = \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 + R^*]. \tag{3.25}$$

In order to obtain the Euler–Lagrange equations in Hamiltonian form, we introduce

$$P_{ab} = \partial L/\partial\dot{\gamma}_{ab} \tag{3.26}$$

and define the Hamiltonian function by

$$H = (\partial L/\partial\dot{\gamma}_{ab})\dot{\gamma}_{ab} - L. \tag{3.27}$$

Since L is homogeneous of degree two in $\dot{\gamma}_{ab}$, we find that the Hamiltonian function is given by

$$H = \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 - R^*], \tag{3.28}$$

and the Euler–Lagrange equations in Hamiltonian form are

$$\dot{\gamma}_{ab} = \partial H/\partial P_{ab}, \quad \dot{P}_{ab} = -\partial H/\partial \gamma_{ab}. \tag{3.29}$$

Since H does not depend explicitly on t, (3.28) has a constant value h for a solution of (3.29), that is,

$$H = \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 - R^*] = h \tag{3.30}$$

is the energy integral. A glance at (3.1) shows that

$$h = 0. \tag{3.31}$$

This is the seventh field equation. Equations (3.2) are integrals of the other equations as Taub proves in Ref. 5. Therefore, the Taub equations can be written as

$$\delta \int_{t_1}^{t_2} \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 + R^*]\, dt = 0, \tag{3.32}$$

$$K_f{}^g C^f{}_{ga} - K_a{}^f C^g{}_{gf} = 0, \tag{3.33}$$

$$H = \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 - R^*] = 0. \tag{3.34}$$

4. VARIATIONAL PRINCIPLE FOR INCOHERENT MATTER

We now consider the Einstein field equations with incoherent matter. Using (2.21), (2.22), (2.23), (2.25), and (2.26), we find that

$$R_{00} - \tfrac{1}{2}R = [K_f{}^g K_g{}^f - (K_f{}^f)^2 - R^*] = -\kappa\rho(u)^2, \tag{4.1}$$

$$R_{a0} = K_f{}^g C^f{}_{ga} - K_a{}^f C^g{}_{gf} = -\kappa\rho u u_a, \tag{4.2}$$

$$R_a{}^b - \tfrac{1}{2}R\delta_a{}^b = (K_a{}^b)^\cdot - (K_f{}^f)^\cdot \delta_a{}^b + (K_f{}^f)K_a{}^b - \tfrac{1}{2}[K_f{}^g K_g{}^f + (K_f{}^f)^2]\delta_a{}^b, \tag{4.3}$$

$$R^*{}_a{}^b - \tfrac{1}{2}R^*\delta_a{}^b = -\kappa\rho u_a u^b.$$

These are the Heckmann–Schücking equations[6] written in a slightly different form. We have, in addition,

$$(u)^2 + u_a u^a = 1, \tag{4.4}$$

$$u\dot{u}_a = C^f{}_{ga} u^g u_f, \tag{4.5}$$

$$(\rho u)^\cdot + (\rho u)[K_f{}^f + (C^g{}_{gf}u^f)/(1 - u_a u^a)^{\frac{1}{2}}] = 0. \tag{4.6}$$

Our problem is now to find the Lagrangian for the Heckmann–Schücking equations. Examining (4.6), we discover that the term

$$C^g{}_{gf}u^f \tag{4.7}$$

contains the vector

$$C^g{}_{gf} \tag{4.8}$$

obtained by contraction over two indices from the structure constant tensor of G_3. The term (4.7) vanishes if (4.8) vanishes. It is natural, therefore, to divide the 3-dimensional Lie groups, their Lie algebras, that is, into two different classes according to the vanishing or nonvanishing of the vector (4.8). These classes are the following:

$$\text{Class I}: \; C^g{}_{ga} = 0, \quad a = 1, 2, 3, \tag{4.9}$$

$$\text{Class II}: \; C^g{}_{ga} \neq 0, \quad a = 1, 2, 3. \tag{4.10}$$

Class I contains the groups of the following Bianchi types:

$$\text{Class I: Type I, II, VIII, and IX.} \tag{4.11}$$

The structure of the Class II algebras is given by

$$[X_1, X_2] = 0, \quad [X_A, X_3] = C^B{}_A X_B, \quad A, B, \cdots = 1, 2, \tag{4.12}$$

or, alternatively,

$$d\omega^1 = -C^1{}_A \omega^A \wedge \omega^3, \quad d\omega^2 = -C^2{}_A \omega^A \wedge \omega^3, \quad d\omega^3 = 0. \tag{4.13}$$

Therefore, the Class II algebras are given by the different normal forms of the 2×2 real matrices $C^B{}_A$ with nonvanishing trace. These are[9]

$$\text{Class II: Type III, IV, V, VI, VII.} \tag{4.14}$$

We write our variational principle for the Class I groups only. We can integrate (4.6) in this case to

$$\rho u = l/\gamma^{\frac{1}{2}} \geq 0, \tag{4.15}$$

where l is a constant. Writing (4.4) as

$$u = (1 - u_a u^a)^{\frac{1}{2}}, \tag{4.16}$$

we obtain from (4.15)

$$\rho = l/[\gamma(1 - u_a u^a)]^{\frac{1}{2}} \tag{4.17}$$

and

$$\rho(u)^2 = l[(1 - u_a u^a)/\gamma]^{\frac{1}{2}}. \tag{4.18}$$

We now claim that the Lagrangian of the Heckmann–Schücking equations for Class I groups is given by

$$L = \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 + R^*] - 2\kappa l(1 - u_a u^a)^{\frac{1}{2}}, \tag{4.19}$$

and the Hamiltonian is

$$H = \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 - R^*] + 2\kappa l(1 - u_a u^a)^{\frac{1}{2}}. \tag{4.20}$$

We obtain the Heckmann–Schücking equations as

$$\delta \int_{t_1}^{t_2} \{\gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 + R^*] - 2\kappa l(1 - u_f u^f)^{\frac{1}{2}}\}\, dt = 0, \tag{4.21}$$

$$K_f{}^g C^f{}_{ga} = -\kappa(l/\gamma^{\frac{1}{2}})u_a, \quad a = 1, 2, 3, \tag{4.22}$$

$$H = \gamma^{\frac{1}{2}}[K_f{}^g K_g{}^f - (K_f{}^f)^2 - R^*] + 2\kappa l(1 - u_f u^f)^{\frac{1}{2}} = h = 0, \tag{4.23}$$

and

$$\dot{u}_a = \frac{C^f{}_{ga} u^g u_f}{(1 - u_f u^f)^{\frac{1}{2}}}, \quad a = 1, 2, 3, \quad C^g{}_{ga} = 0!. \quad (4.24)$$

To prove this assertion, we compute the derivative $dF/d\alpha$ of

$$F(\alpha) = -2\kappa l \int_{t_1}^{t_2} (1 - u_a u^a)^{\frac{1}{2}}\, dt \quad \text{for} \quad \alpha = 0.$$

We imagine that the functions $\gamma_{ab}(\alpha; t)$ are substituted for $\gamma_{ab}[t]$:

$$\frac{dF}{d\alpha} = -\kappa l \int_{t_1}^{t_2} \frac{u^a u^b}{(1 - u_f u^f)^{\frac{1}{2}}} \gamma'_{ab}\, dt. \quad (4.25)$$

Using our earlier results, we see that

$$-(\gamma^{\frac{1}{2}})[R^{ab} - \tfrac{1}{2} R \gamma^{ab}] - [\kappa l/(1 - u_f u^f)^{\frac{1}{2}}] u^a u^b = 0. \quad (4.26)$$

Dividing by $-\gamma^{\frac{1}{2}}$ and using (4.17), we find that (4.26) is equivalent to (4.3). It follows, exactly as before, that the Hamiltonian function (4.20) has to be constant for the solutions. Dividing (4.20) by $2\gamma^{\frac{1}{2}}$, using (4.18) and (4.1), we see that the energy constant has to be zero. Equations (4.22) and (4.24) are consistent with the other equations (see Ref. 6).

5. SOME REMARKS

Examining Eqs. (3.32)–(3.34), one sees that the variational principle has its full power in the vacuum case. One would treat the vacuum problem as a mechanical problem defined by the Lagrangian function:

$$L = \gamma^{\frac{1}{2}} [K_f{}^g K_g{}^f - (K_f{}^f)^2 + R^*], \quad (5.1)$$

and reduce it with the help of the integrals

$$K_f{}^g C^f{}_{ga} - K_a{}^f C^g{}_{gf} = 0, \quad a = 1, 2, 3. \quad (5.2)$$

In case of dust, the situation is different. The Class I groups are preferred because (4.6) is then integrable. Furthermore, the power of (4.21)–(4.23) is limited by (4.24) in general. We see in the last section of this paper that, in the case of

$$ds^2 = dt^2 + A(\omega^1)^2 + B(\omega^2)^2 + C(\omega^3)^2 + 2D\omega^2\omega^3, \quad (5.3)$$

(4.24) is trivial and the method retains its full power and simplicity. In order to obtain the *general* Type VIII and IX models (the Type I and II models do not have rotation), one develops the above formulas for the line element

$$ds^2 = (dt + p_f \omega^f)^2 + \gamma_{ab}(t) \omega^a \omega^b, \quad a, b, f, \cdots = 1, 2, 3, \quad (5.4)$$

with

$$\dot{p}_a = 0, \quad a = 1, 2, 3, \quad (5.5)$$

which replaces (4.24) as the geodesic condition. The expressions corresponding to (4.21)–(4.23) are naturally more involved.

Another remark refers to the Bianchi Type II group given by

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = 0, \quad d\omega^3 = 0. \quad (5.6)$$

We claim that there is no rotating solution for (5.6). The proof is trivial. From (4.22), we obtain

$$K_f{}^g C^f{}_{g1} = 0 = \kappa (l/\gamma^{\frac{1}{2}}) u_1,$$

that is,

$$u_1 = 0. \quad (5.7)$$

Due to the special form of the structure constant tensor, it follows that

$$C^f{}_{ab} u_f \equiv C^1{}_{ab} u_1 = 0. \quad (5.8)$$

Using (4.24) and (5.8), one sees that

$$\dot{u}_a = 0, \quad a = 1, 2, 3. \quad (5.9)$$

Since the components of the rotation tensor are given by

$$\omega_{ac} = -\tfrac{1}{2} C^f{}_{ab} u_f, \quad \omega_{a0} = -\tfrac{1}{2} \dot{u}_a$$

[see (2.43)], we obtain

$$\omega_{ab} = 0, \quad \omega_{a0} = 0, \quad (5.10)$$

as claimed. Rotating solution for Class I groups is, therefore, possible only with Bianchi Type VIII and IX groups. We now go over to more serious applications.

6. A CLASS OF SOLUTIONS

We consider the line element

$$ds^2 = dt^2 - (A\omega^1)^2 - (B\omega^2)^2 - (B\omega^3)^2 \quad (6.1)$$

with each of the four different Class I groups; that is, with

$$d\omega^1 = 0, \quad d\omega^2 = 0, \quad d\omega^3 = 0, \quad (6.2)$$

or

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = 0, \quad d\omega^3 = 0, \quad (6.3)$$

or

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^3 \wedge \omega^1, \quad d\omega^3 = -\omega^1 \wedge \omega^2, \quad (6.4)$$

or

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^3 \wedge \omega^1, \quad d\omega^3 = -\omega^1 \wedge \omega^2, \quad (6.5)$$

respectively. It is *not known* by me whether or not the cases with (6.3) and (6.4) are in the literature, but (6.1) with (6.5) has been discussed. With vacuum this is the Taub solution and with incoherent matter it is discussed by Behr.[10] The case (6.1) with (6.2) is fully integrated by Schücking. Our aim is to compute the Hamiltonian function of these cases and write the field equations in Hamiltonian form and *regularize* these equations, in the way that Sundman regularized the 3-body problem of the celestial mechanics. The idea is as follows: One introduces by a suitable transformation

$$t = t(\tau) \quad \text{or} \quad \tau = \tau(t),$$

a new independent variable τ such that our field equations, as a system of first order ordinary differential equations with respect to τ, should be *analytic*. A system

$$x'_\kappa = f_\kappa(x_l), \quad \kappa, l, \cdots = 1, 2, \cdots, n,$$

is called analytic if the functions f_κ as functions of x_l are analytic. Having done that successfully, one considers the problem solved since everything else can be done by computers and by the application of the qualitative theory of differential equations.[11]

To demonstrate all of this by an example, we proceed with our problem. From (4.22), it follows that

$$u = 1, \quad u_a = 0, \quad a = 1, 2, 3, \tag{6.6}$$

and Eqs. (4.24) are trivially satisfied. The Lagrangian and Hamiltonian functions read as

$$L = -4\dot{A}\dot{B}B - 2A\dot{B}^2 + AB^2R^* - 2\kappa l \tag{6.7}$$

and

$$H = -4\dot{A}\dot{B}B - 2A\dot{B}^2 - AB^2R^* + 2\kappa l, \tag{6.8}$$

respectively, where the relevant Ricci scalars are

for Eq. (6.2)

$$R^* = 0, \tag{6.9}$$

for Eq. (6.3)

$$R^* = -A^2/2B^4, \tag{6.10}$$

for Eq. (6.4)

$$R^* = -(A^2 + 4B^2)/2B^4, \tag{6.11}$$

for Eq. (6.5)

$$R^* = -(A^2 - 4B^2)/2B^4. \tag{6.12}$$

We treat the case (6.12), that is, (6.1) with (6.5). The other cases can be obtained by making suitable changes.

The Hamiltonian function for (6.1) with (6.5) is given by

$$H = -4\dot{A}\dot{B}B - 2A\dot{B}^2 + (A^3/2B^2) - 2A + 2\kappa l. \tag{6.13}$$

We write the field equations in Hamiltonian form, that is, we define

$$P = \frac{\partial L}{\partial \dot{A}} = -4B\dot{B}, \quad Q = \frac{\partial L}{\partial \dot{B}} = -4\dot{A}B - 4A\dot{B}. \tag{6.14}$$

Solving (6.14) for $\dot{A}$ and $\dot{B}$, we obtain

$$\dot{A} = (AP/4B^2) - (Q/4B), \quad \dot{B} = -P/4B. \tag{6.15}$$

Substituting into (6.13), we obtain

$$H = (AP^2/8B^2) - (PQ/4B) + (A^3/2B^2) - 2A + 2\kappa l, \tag{6.16}$$

and the field equations in Hamiltonian form are

$$\dot{A} = \partial H/\partial P, \quad \dot{B} = \partial H/\partial Q, \quad \dot{P} = -\partial H/\partial A, \quad \dot{Q} = -\partial H/\partial B. \tag{6.17}$$

The first two equations are (6.15), and the second two are given by

$$\dot{P} = -(P^2/8B^2) - (3A^2/2B^2) + 2,$$
$$\dot{Q} = (AP^2/4B^3) - (PQ/4B^2) + A^3/B^3. \tag{6.18}$$

The energy integral reads as

$$H \equiv (AP^2/8B^2) - (PQ/4B) + (A^3/2B^2) - 2A + 2\kappa l = 0. \tag{6.19}$$

The form of (6.15) and (6.18) strongly suggests the introduction of the new variables

$$x = A/B, \quad y = P/B, \quad z = Q/B, \tag{6.20}$$

or

$$A = xB, \quad P = yB, \quad Q = zB. \tag{6.21}$$

Then, the equations read as

$$\dot{x}B = \tfrac{1}{2}xy - \tfrac{1}{4}z, \tag{6.22}$$

$$\dot{B}B = -\tfrac{1}{4}yB, \tag{6.23}$$

$$\dot{y}B = \tfrac{1}{8}y^2 - \tfrac{3}{2}x^2 + 2, \tag{6.24}$$

$$\dot{z}B = x(x^2 + \tfrac{1}{4}y^2), \tag{6.25}$$

and

$$B[\tfrac{1}{8}xy^2 - \tfrac{1}{4}yz + \tfrac{1}{2}x^3 - 2x] + 2\kappa l = 0. \tag{6.26}$$

Introducing a new independent variable τ by

$$\frac{df}{d\tau} \equiv f' = \dot{f}B, \tag{6.27}$$

our equations become

$$x' = \tfrac{1}{2}xy - \tfrac{1}{4}z, \tag{6.28}$$

$$B' = -\tfrac{1}{4}yB, \tag{6.29}$$

$$y' = \tfrac{1}{8}y^2 - \tfrac{3}{2}x^2 + 2, \tag{6.30}$$

$$z' = x(x^2 + \tfrac{1}{4}y^2), \tag{6.31}$$

where Eq. (6.29) is a consequence of (6.26), (6.28), (6.30), and (6.31). A more elegant way to do this would be to reverse the order of the operations. One should carry out the time transformation first with the help of a canonical transformation (see Ref. 4, 35). It is then obvious that one retains an energy integral and, therefore, one can leave (6.29) aside. And, as a second step, one would go over to the ratios.

We want to give our method for solving the problem: We integrate the *analytic* system

$$x' = \tfrac{1}{2}xy - \tfrac{1}{4}z, \quad y' = \tfrac{1}{8}y^2 - \tfrac{3}{2}x^2 + 2, \quad z' = x(x^2 + \tfrac{1}{4}y^2); \tag{6.32}$$

we compute B from (6.26)

$$B = 4\kappa l/(-\tfrac{1}{4}xy^2 + \tfrac{1}{2}yz - x^3 + 4x); \tag{6.33}$$

we compute A from

$$A = xB. \tag{6.34}$$

The cosmic time t is computed from

$$t = \int_{\tau_0}^{\tau} \frac{4\kappa l}{-\tfrac{1}{4}xy^2 + \tfrac{1}{2}yz - x^3 + 4x}\, d\sigma. \tag{6.35}$$

This method solves our problem in a way similar to Sundman's solution of the 3-body problem of the celestial mechanics.

The best way of visualizing (6.32) is to go into a 3-dimensional Euclidean space E_3 with the coordinates x, y, and z. The right-hand sides of (6.32) are the components of an analytic vector field V over E_3. V is nowhere singular; that is, the components of V vanish nowhere on E_3 simultaneously.

Through any point $P_0 = (x_0, y_0, z_0)$ of E_3, one can draw with the help of a computer one and only one integral curve C of (6.32). We then find, corresponding to each such line, a universe following the rest of our method. Singularities of the universe occur, for example, where C goes through the yz plane; that is, $x = 0$ since the geometrical meaning of x is the ratio of the axes of the universe. To find the answers to the arising questions, one should study Nemytskii and Stepanov.[11] Some numerical calculations will be made in a later paper. As a curiosity, we compute the Friedmann cosmos. Assuming

$$x = 1, \tag{6.36}$$

then Eqs. (6.32) reduce to

$$z = 2y, \tag{6.37}$$

$$y' = \tfrac{1}{2}(1 + \tfrac{1}{4}y^2), \tag{6.38}$$

which integrate to

$$y = 2tg\tfrac{1}{4}(\tau - \tau_0). \tag{6.39}$$

Then, (6.33) reads

$$B = \tfrac{4}{3}\kappa l \cos^2 \tfrac{1}{4}(\tau - \tau_0) \tag{6.40}$$

and

$$t = \tfrac{4}{3}\kappa l[\tfrac{1}{2}(\tau - \tau_0) + \sin \tfrac{1}{2}(\tau - \tau_0)]. \tag{6.41}$$

We now list the equations for the other cases:

(6.1) with (6.2),

$$x' = \tfrac{1}{2}xy - \tfrac{1}{4}z, \quad y' = \tfrac{1}{8}y^2, \quad z' = \tfrac{1}{4}xy^2,$$
$$B = 16\kappa l/[y(2z - xy)], \quad A = xB, \quad t = \int_{\tau_0}^{\tau} B\, d\sigma; \tag{6.42}$$

(6.1) with (6.3),

$$x' = \tfrac{1}{2}xy - \tfrac{1}{4}z, \quad y' = \tfrac{1}{8}y^2 - \tfrac{}{2}x^2, \quad z' = x(x^2 + \tfrac{1}{4}y^2),$$
$$B = \frac{4\kappa l}{-\tfrac{1}{4}xy^2 + \tfrac{1}{2}yz - x^3}, \quad A = xB, \quad t = \int_{\tau_0}^{\tau} B\, d\sigma; \tag{6.43}$$

(6.1) with (6.4),

$$x' = \tfrac{1}{2}xy - \tfrac{1}{4}z, \quad y' = \tfrac{1}{8}y^2 - \tfrac{3}{2}x^2 - 2,$$
$$z' = x(x^2 + \tfrac{1}{4}y^2),$$
$$B = \frac{4\kappa l}{-\tfrac{1}{4}xy^2 + \tfrac{1}{2}yz - x^3 - 4x}, \quad A = xB, \tag{6.44}$$
$$t = \int_{\tau_0}^{\tau} B\, d\sigma.$$

As a curiosity, we remark that (6.42) can be integrated in closed form:

$$x = \frac{\alpha}{3(\tau_0 - \tau)^3} + \beta, \quad y = \frac{8}{\tau_0 - \tau},$$
$$z = \frac{4}{\tau_0 - \tau}\left[\frac{\alpha}{3(\tau_0 - \tau)^3} + 4\beta\right], \tag{6.45}$$
$$B = \frac{\kappa l}{12\beta}(\tau_0 - \tau)^2, \quad A = xB, \quad t = -\frac{\kappa l}{36\beta}(\tau_0 - \tau)^3,$$

where τ_0, α, and β are constants of integration. The corresponding solution is a Schücking solution.

7. THE ROTATING UNIVERSES

There is a challenging problem in the relativistic cosmology: Consider the line element

$$ds^2 = dt^2 + A(\omega^1)^2 + B(\omega^2)^2 + C(\omega^3)^2 + 2D\omega^2\omega^3, \tag{7.1}$$

where A, B, C, and D are functions of t only and the differential forms ω^1, ω^2, and ω^3 satisfy either

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^3 \wedge \omega^1, \quad d\omega^3 = -\omega^1 \wedge \omega^2 \tag{7.2}$$

or

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^3 \wedge \omega^1, \quad d\omega^3 = -\omega^1 \wedge \omega^2. \tag{7.3}$$

We find A, B, C, and D such that (7.1) satisfies Einstein's field equations with incoherent matter and such that the expansion and the rotation of the matter do not vanish.

This problem is challenging not because the astronomers discovered the rotation of the universe, but because (7.1) with (7.3) probably gives the simplest finite model where the "Weltsubstrat" has the most general motion, namely, translation, rotation, expansion, and shear. For the sake of definiteness, we restrict ourselves to (7.1) with (7.3). We see that

$$\gamma_{ab} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & D \\ 0 & D & C \end{pmatrix},$$

$$\gamma^{ab} = \begin{pmatrix} \frac{1}{A} & 0 & 0 \\ 0 & \frac{C}{BC - D^2} & -\frac{D}{BC - D^2} \\ 0 & -\frac{D}{BC - D^2} & \frac{B}{BC - D^2} \end{pmatrix}, \quad (7.4)$$

and, therefore,

$$K_a{}^b = \begin{pmatrix} \frac{\dot{A}}{2A} & 0 & 0 \\ 0 & \frac{\dot{B}C - D\dot{D}}{2(BC - D^2)} & \frac{B\dot{D} - \dot{B}D}{2(BC - D^2)} \\ 0 & \frac{C\dot{D} - \dot{C}D}{2(BC - D^2)} & \frac{B\dot{C} - D\dot{D}}{2(BC - D^2)} \end{pmatrix}, \quad (7.5)$$

where a is the row and b is the column index. From (4.22) and (7.5), it follows that

$$\frac{(B - C)\dot{D} - (\dot{B} - \dot{C})D}{2(BC - D^2)} = -\kappa \frac{l}{\gamma^{\frac{1}{2}}} u_1,$$

$$u_2 = 0, \quad u_3 = 0, \quad (7.6)$$

and from (4.24), we obtain

$$\dot{u}_1 = 0. \quad (7.7)$$

Therefore,

$$u_\alpha = \left[\left(1 - \frac{V^2}{A}\right)^{\frac{1}{2}}, V, 0, 0\right], \quad (7.8)$$

where V is a constant. One might mention that the only nonvanishing component of the rotation tensor [see (2.43)] is given by

$$\omega_{23} = -\tfrac{1}{2}V, \quad (7.9)$$

and the length of the vector of rotation W defined by

$$W^\alpha = \tfrac{1}{2}\eta^{\alpha\beta\gamma\delta}\omega_{\beta\gamma}u_\delta \quad (7.10)$$

is given by

$$g(W, W) = A(\tfrac{1}{2}V)^2. \quad (7.11)$$

We write (7.6) for later references in the following form:

$$A[(B - C)\dot{D} - (\dot{B} - \dot{C})D] = 2\kappa l V \gamma^{\frac{1}{2}}, \quad (7.12)$$

where

$$\gamma = |A(BC - D^2)|. \quad (7.13)$$

One easily computes that

$$L = \gamma^{\frac{1}{2}}\left(-\frac{\dot{A}(BC - D^2)^{\cdot}}{2A(BC - D^2)} - \frac{A(\dot{B}\dot{C} - \dot{D}^2)}{2A(BC - D^2)} + R^*\right) - 2\kappa l\left(1 - \frac{V^2}{A}\right)^{\frac{1}{2}}. \quad (7.14)$$

The Ricci scalar of the group space is given by

$$R^* = \frac{2(A^2 + B^2 + C^2) - (A + B + C)^2 + 4D^2}{2A(BC - D^2)}. \quad (7.15)$$

There are two problems: (a) reduction of the mechanical system, defined by the Lagrangian function (7.14), with the help of the first integral (7.12); (b) regularization of the reduced system.

A. Reduction of the Mechanical System

The reduction of a system is a standard problem in the mechanics, and we found its solution following standard methods. Therefore, we give the results only. Consider the functions x, y, z, and w defined by

$$x = -A, \quad y = -\tfrac{1}{2}\{B + C + [4D^2 + (B - C)^2]^{\frac{1}{2}}\},$$
$$z = -\tfrac{1}{2}\{B + C - [4D^2 + (B - C)^2]^{\frac{1}{2}}\}, \quad (7.16)$$
$$w = \arctan (B - C/2D),$$

or the inverse transformations

$$A = -x, \quad B = \tfrac{1}{2}(y + z) - \tfrac{1}{2}(y - z)\sin w,$$
$$C = \tfrac{1}{2}(y + z) + \tfrac{1}{2}(y - z)\sin w, \quad (7.17)$$
$$D = -\tfrac{1}{2}(y - z)\cos w.$$

We show that (7.17) reduces our system defined by (7.14) and (7.12). We first compute the new form of (7.12). One sees that

$$yz = BC - D^2; \quad (7.18)$$

therefore, $xyz = -A(BC - D^2)$ and

$$\gamma^{\frac{1}{2}} = (xyz)^{\frac{1}{2}}. \quad (7.19)$$

One easily computes that

$$A[(B - C)\dot{D} - (\dot{B} - \dot{C})] = \tfrac{1}{2}x(y - z)^2\dot{w};$$

therefore, (7.12) reads as

$$\dot{w} = \frac{4\kappa l V}{x(y - z)^2}(xyz)^{\frac{1}{2}}. \quad (7.20)$$

It does not contain w! We now compute the Lagrangian function (7.14) in the new variables and find that it does not contain w. Using (7.18), we see that

$$-\frac{\dot{A}}{2A}\frac{(BC - D^2)^{\cdot}}{BC - D^2} = -\frac{1}{2}\frac{\dot{x}}{x}\left(\frac{\dot{y}}{y} + \frac{\dot{z}}{z}\right). \quad (7.21)$$

A straightforward calculation shows that

$$\dot{B}\dot{C} - \dot{D}^2 = \dot{y}\dot{z} - \left(\frac{y-z}{z}\right)^2(\dot{w})^2 = \dot{y}\dot{z} - \frac{4\kappa^2 l^2 V^2}{x(y-z)^2} yz;$$

therefore,

$$-\frac{A}{2A}\frac{(\dot{B}\dot{C} - \dot{D}^2)}{(BC - D^2)} = -\frac{1}{2}\frac{\dot{y}}{y}\frac{\dot{z}}{z} + \frac{2x^2 l^2 V^2}{x(y-z)^2}. \quad (7.22)$$

Another trivial calculation shows that

$$R^* = -\frac{2(x^2+y^2+z^2) - (x+y+z)^2}{2xyz}. \quad (7.23)$$

From (7.14), (7.19), (7.21), (7.22), and (7.23) it then follows that

$$\begin{aligned} L = -(xyz)^{\frac{1}{2}}\Bigg(&\frac{1}{2}\frac{\dot{x}}{x}\frac{\dot{y}}{y} + \frac{1}{2}\frac{\dot{y}}{y}\frac{\dot{z}}{z} + \frac{1}{2}\frac{\dot{z}}{z}\frac{\dot{x}}{x} \\ &+ \frac{2(x^2+y^2+z^2) - (x+y+z)^2}{2xyz} - \frac{2\kappa^2 l^2 V^2}{x(y-z)^2}\Bigg) \\ &- 2\kappa l\left(1 + \frac{V^2}{x}\right)^{\frac{1}{2}}. \end{aligned} \quad (7.24)$$

The Lagrangian function has been first computed by Gödel.[1] (Also, see Ref. 12.) Defining

$$p = \frac{\partial L}{\partial \dot{x}}, \quad q = \frac{\partial L}{\partial \dot{y}}, \quad r = \frac{\partial L}{\partial \dot{z}}, \quad (7.25)$$

we compute the Hamiltonian function

$$\begin{aligned} H = \frac{1}{2(xyz)^{\frac{1}{2}}}\{&2[(xp) + (yq)^2 + (zr)^2] \\ &- (xp + yq + zr)^2 \\ &+ 2(x^2+y^2+z^2) - (x+y+z)^2\} \\ &- \frac{2\kappa^2 l^2 V^2}{x(y-z)^2}(xyz)^{\frac{1}{2}} + 2\kappa l\left(1 + \frac{V^2}{x}\right)^{\frac{1}{2}}. \end{aligned} \quad (7.26)$$

The field equations are

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{y} = \frac{\partial H}{\partial q}, \quad \dot{z} = \frac{\partial H}{\partial r},$$

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{q} = -\frac{\partial H}{\partial y}, \quad \dot{r} = -\frac{\partial H}{\partial z}, \quad (7.27)$$

where H is given by (7.26).

The method for finding a rotating universe is as follows: Find a solution of (7.27) for which

$$H = 0. \quad (7.28)$$

Then we obtain w from (7.20) by integration. Compute the components of the metric from (7.17). Examining the form of (7.17), one sees that seeking a solution via the ansatz

$$\gamma_{ab} = \begin{pmatrix} a & 0 & 0 \\ 0 & b + c\sin\alpha & c\cos\alpha \\ 0 & c\cos\alpha & b - c\sin\alpha \end{pmatrix},$$

where a, b, c, and α are unknown functions of time, is a naive but well-founded approach. We now consider our second problem.

B. Regularization of the Reduced System

We formulate this problem as follows: Introduce a new independent variable by a suitable transformation of t such that (7.27) is transformed into an *analytic* system. Examining (7.26), one had the strong impression that *the problem of the rotating universe can be solved by regularization.* One sees several ways and one has several suggestions; the strongest one is probably to study Siegel's book.[4]

Note added in proof: I am indebted to G. F. R. Ellis for bringing to my attention the following two papers,

G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. **12,** 108 (1969),

S. Hawking, Monthly Notices Roy. Astron. Soc. **142,** 129 (1969),

and for the remark that there are two additional groups of Class I, namely, a special Type VI and a special Type VII group characterized by

$$d\omega^1 = \omega^3 \wedge \omega^1, \quad d\omega^2 = \omega^2 \wedge \omega^3, \quad d\omega^3 = 0$$

and

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad d\omega^2 = \omega^3 \wedge \omega^1, \quad d\omega^3 = 0,$$

respectively.

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